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LETTER TO THE EDITOR

The n-dimensional κ-Poincaré algebra and group

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Abstract. The n-dimensional κ-Poincaré algebra and its global counterpart are described.

Recently an interesting deformation of Poincaré algebra—the so-called κ-Poincaré algebra—has been constructed [1] (see also [2]). The global counterpart of the κ-Poincaré algebra was constructed by Zakrzewski [3]. In the present letter we generalize this discussion to the n-dimensional case. The straightforward generalization of the four-dimensional case gives us the following κ-Poincaré algebra in n dimensions (with M_{ij} being rotations while M_{i0} are boosts, $i, j = 1, 2, \ldots, n-1$):

$$
\begin{aligned}
&[M_{ij}, M_{kl}] = \mathrm{i}(\delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} - \delta_{jk} M_{il}) \qquad [M_{ij}, P_k] = \mathrm{i}(\delta_{ik} P_j - \delta_{jk} P_i) \\
&[M_{ij}, P_0] = 0 \qquad [M_{i0}, P_k] = \mathrm{i}\kappa\delta_{ik} \sinh\left(\frac{P_0}{\kappa}\right) \qquad [M_{i0}, P_0] = \mathrm{i}P_i \\
&[M_{ij}, M_{k0}] = \mathrm{i}(\delta_{ik} M_{j0} - \delta_{jk} M_{i0}) \\
&[M_{i0}, M_{j0}] = -\mathrm{i}\left[M_{ij} \cosh\left(\frac{P_0}{\kappa}\right) - \frac{1}{4\kappa^2}\left(M_{ij} \boldsymbol{P}^2 + \sum_{k=1}^{n-1} P_i M_{jk} P_k - \sum_{k=1}^{n-1} P_j M_{ik} P_k\right)\right] \\
&[P_\mu, P_\nu] = 0
\end{aligned}
\tag{1}
$$

where $\mu, \nu = 0, 1, \ldots, n-1$.

The comultiplication is defined as follows:

$$
\begin{aligned}
&\Delta P_0 = P_0 \otimes I + I \otimes P_0 \qquad \Delta P_i = P_i \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes P_i \\
&\Delta M_{ij} = M_{ij} \otimes I + I \otimes M_{ij} \\
&\Delta M_{i0} = M_{i0} \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes M_{i0} + \frac{1}{2\kappa} \sum_{j=1}^{n-1} P_j \otimes M_{ij} \exp\left(\frac{P_0}{2\kappa}\right) \\
&\qquad - \frac{1}{2\kappa} \sum_{j=1}^{n-1} \exp\left(-\frac{P_0}{2\kappa}\right) M_{ij} \otimes P_j
\end{aligned}
\tag{2}
$$

while the antipode is given by

$$
S(P_\mu) = -P_\mu \qquad S(M_{ij}) = -M_{ij} \qquad S(M_{i0}) = -M_{i0} + \frac{i(n-1)}{2\kappa} P_i . \tag{3}
$$

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The classical n-dimensional Poincaré group consists of matrices of the form

$$g = (g^i_j)_{i,j=0,\ldots,n} = \begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix} \tag{4}$$

where $\Lambda = (\Lambda^\mu_\nu)$ belongs to the Lorentz group and $v = (v^\mu) \in \mathbb{R}^n$ ($\mu, \nu = 0, 1, \ldots, n-1$).

In order to quantize this group we first define, using co-product (2), the co-commutator

$$\delta = (\Delta - \sigma \circ \Delta) \qquad \mathrm{mod}\frac{1}{\kappa}\,. \tag{5}$$

Here σ is the flip operator $\sigma : a \otimes b \to b \otimes a$.

We have

$$\begin{aligned} &\delta(M_{ij}) = 0 \qquad \delta(P_0) = 0 \qquad \delta(P_i) = \frac{1}{\kappa}(P_i \wedge P_0) \\ &\delta(M_{i0}) = \frac{1}{\kappa}\left(M_{i0} \wedge P_0 + \sum_{k=1}^{n-1} P_k \wedge M_{ik}\right). \end{aligned} \tag{6}$$

Again, as in four dimensions [3], δ is a co-boundary

$$\delta(x) = \mathrm{ad}_x r \qquad r = \frac{1}{\kappa}\sum_{k=1}^{n-1} M_{k0} \wedge P_k\,. \tag{7}$$

A calculation of Shouten bracket of r with itself yields

$$[r, r] = \frac{1}{\kappa} r \wedge P_0 - \frac{1}{\kappa^2}\sum_{k,l=1}^{n-1}(M_{kl} \otimes P_k \otimes P_l + P_k \otimes M_{kl} \otimes P_l + P_k \otimes P_l \otimes M_{kl}) \tag{8}$$

which is invariant; therefore r is a classical r-matrix. It is easy to check that in only three dimensions we can improve the r-matrix (by adding the symmetric term) in such a way that the new r-matrix is the solution to the classical Yang–Baxter equation; it reads

$$\begin{aligned} r = \frac{1}{\kappa}\bigg(&\sum_{k=1}^{2} M_{k0} \wedge P_k + \mathrm{i}M_{20} \otimes P_1 + \mathrm{i}P_1 \otimes M_{20} - \mathrm{i}M_{10} \otimes P_2 \\ &- \mathrm{i}P_2 \otimes M_{10} + \mathrm{i}M_{12} \otimes P_0 + \mathrm{i}P_0 \otimes M_{12}\bigg)\,. \end{aligned} \tag{9}$$

By calculating the Poisson bivector $\Pi(g) = gr - rg$, we determine the Poisson brackets of the coordinate functions on the Poincaré group:

$$\begin{aligned} &\{\Lambda^\mu_\nu, \Lambda^\alpha_\beta\} = 0 \qquad \{v^k, v^0\} = \frac{1}{\kappa}v^k \qquad \{\Lambda^\mu_\nu, v^0\} = \frac{1}{\kappa}(\Lambda^\mu_0\Lambda^0_\nu - \delta^{\mu 0}\delta_{\nu 0}) \\ &\{\Lambda^m_0, v^r\} = \frac{1}{\kappa}(\delta_{mr} - \delta_{mr}\Lambda^0_0 + \Lambda^m_0\Lambda^r_0) \qquad \{\Lambda^0_0, v^r\} = \frac{1}{\kappa}(\Lambda^0_0\Lambda^r_0 - \Lambda^r_0) \\ &\{\Lambda^0_m, v^r\} = \frac{1}{\kappa}(\Lambda^0_0\Lambda^r_m - \Lambda^r_m) \qquad \{\Lambda^m_l, v^r\} = \frac{1}{\kappa}(\Lambda^m_0\Lambda^r_l - \Lambda^0_l\delta_{rm})\,. \end{aligned} \tag{10}$$

In order to obtain the corresponding quantum group we consider the universal $*$-algebra with unity generated by self-adjoint elements Λ^{μ}_{ν}, v^{μ} ($\mu, \nu = 0, 1, \ldots, n-1$), subject to the following relations:

$$[\Lambda^{\mu}_{\nu}, \Lambda^{\alpha}_{\beta}] = 0 \qquad [v^{k}, v^{0}] = \frac{\mathrm{i}}{\kappa} v^{k} \qquad [\Lambda^{\mu}_{\nu}, v^{0}] = \frac{1}{\kappa}(\Lambda^{\mu}_{0}\Lambda^{0}_{\nu} - \delta^{\mu 0}\delta_{\nu 0})$$
$$[\Lambda^{m}_{0}, v^{r}] = \frac{\mathrm{i}}{\kappa}(\delta_{mr} - \delta_{mr}\Lambda^{0}_{0} + \Lambda^{m}_{0}\Lambda^{r}_{0}) \qquad [\Lambda^{0}_{0}, v^{r}] = \frac{\mathrm{i}}{\kappa}(\Lambda^{0}_{0}\Lambda^{r}_{0} - \Lambda^{r}_{0}) \tag{11}$$
$$[\Lambda^{0}_{m}, v^{r}] = \frac{\mathrm{i}}{\kappa}(\Lambda^{0}_{0}\Lambda^{r}_{m} - \Lambda^{r}_{m}) \qquad [\Lambda^{m}_{l}, v^{r}] = \frac{\mathrm{i}}{\kappa}(\Lambda^{m}_{0}\Lambda^{r}_{l} - \Lambda^{0}_{l}\delta_{rm}).$$

which are obtained from (10) by making a replacement $\{\ ,\ \} \to 1/\mathrm{i}[\ ,\ ]$. This set of relations is consistent: there is no ordering ambiguity, when 'quantizing' the right-hand side of (10) because of commutativity in the first relation of (11). Moreover, since the standard comultiplication is compatible with Poisson brackets (10), it is also compatible with relations (11). We conclude that the above relations together with the standard comultiplication define a Hopf $*$-algebra.

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